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TECHNICAL REPORT : COMMUNICATIONS

DELAY-LOCK TRACKING OF BINARY SIGNALS

Lockheed
MISSILES and SPACE DIVISION
LOCKHEED AIRCRAFT CORPORATION · SUNNYVALE, CALIF.

NOTICE

WORK CARRIED OUT AS PART OF THE LOCKHEED INDEPENDENT RESEARCH PROGRAM.

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DELAY-LOCK TRACKING OF BINARY SIGNALS

by
J. J. SPILKER, JR.

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A GROUP DIVISION OF LOCKHEED AIRCRAFT CORPORATION
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DELAY-LOCK TRACKING OF BINARY SIGNALS

J. J. Spilker, Jr.,* Member, IRE

Summary

This paper presents the theory of operation and an evaluation of performance of a delay-lock tracking system for binary signals. The delay-lock discriminator is a nonlinear feedback system which employs a form of cross-correlation in the feedback loop. Its function is to estimate continuously the relative delay between a reference signal and a delayed version of that signal which is perturbed with additive noise. Binary maximal-length shift-register sequences are used as the signal because they can easily be regenerated with any desired delay and because they possess desirable autocorrelation functions.

Problems of target search and acquisition are studied. The system performance in the presence of additive Gaussian noise is discussed. Computations are made of the effect of amplitude-limiting the received data on the system noise performance.

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DELAY-LOCK TRACKING OF BINARY SIGNALS

J. J. Spilker, Jr., Member, IRE

I INTRODUCTION

The delay-lock discriminator has been described previously[1,2] as a device for tracking the delay difference between two correlated waveforms. The discriminator is a nonlinear feedback system which employs a form of cross-correlation in the feedback loop. This device tracks the delay of a broadband signal much in the same manner that a phase-lock discriminator tracks the phase of a sinusoidal signal. In this paper, a modified version of the delay-lock discriminator is described which is particularly designed to track binary signals generated from feedback shift-registers. One of the main reasons for the interest in this type of signal is that it is easy to regenerate the signal in the discriminator for use in the cross-correlation operations with any desired amount of delay. Furthermore, certain classes of these sequences possess desirable autocorrelation properties; e.g., the maximal-length linear shift-register sequences[3,4] have "two-level" autocorrelation functions and can be designed to have periods which are long enough so that ambiguities are of no concern.

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[1] J. J. Spilker, Jr., D. T. Magill, "The delay-lock discriminator – an optimum tracking device," Proc. IRE, vol. 49, pp. 1403–1416, September, 1961.

[2] M. R. O'Sullivan, "Tracking systems employing the delay-lock discriminator," IRE Trans., vol. SET-8, pp. 1–7, March, 1962.

[3] B. Elspas, "The theory of autonomous linear sequential switching networks," IRE Trans., vol. CT-6, pp. 45–60, March, 1959.

[4] S. W. Golomb, L. R. Welch, "Nonlinear shift-register sequences," Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California, Memorandum #20-149, October, 1957.

In an actual tracking operation, the binary sequence is applied to an RF carrier and is transmitted to the target, where it is returned to the tracking equipment either as a radar reflection or via a transponder. The RF signal is then demodulated, and the demodulated binary sequence plus additive noise is fed into the delay-lock discriminator to obtain the estimate of the delay between the original and the received sequences. This paper is concerned only with the delay estimation operation after demodulation of the RF carrier.

The analysis of this feedback system is presented in four main parts:

1. Derivation of the system equations
2. Evaluation of noise effects, caused by receiver noise and system self-noise
3. Determination of transient performance – acquisition and pull-out effects
4. Computation of noise effects with amplitude-limiting inputs

II DESCRIPTION OF THE DISCRIMINATOR

A block diagram of the modified delay-lock discriminator is shown in Fig. 1. The received signal originated from an n-stage maximal-length linear feedback shift-register (FSR), and has a period $M\Delta = (2^n - 1)\Delta$, where Δ is the digit width. This binary signal plus additive Gaussian white noise is fed into the cross-correlation network, where a comparison is made with time-displaced versions of the same pseudo-random binary signal as used in the transmitter. As is shown in a later section, the implementation of this cross-correlator can be substantially simplified if the received data is first converted to binary form by means of an amplitude-limiter. Then the multiplication of ± 1 binary digits can be replaced by a modulo 2 adder.

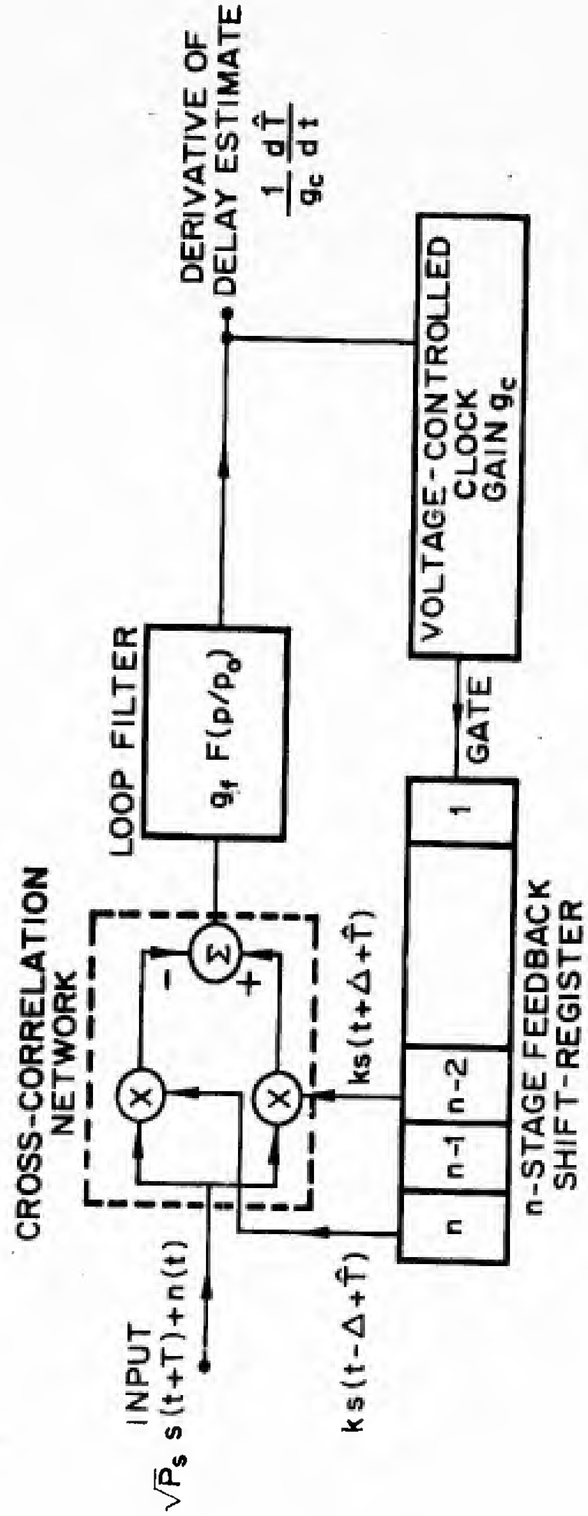

Fig. 1 Delay-lock discriminator for binary shift-register sequences

The output of the cross-correlator network in Fig. 1 is[5]

$$k\delta s\left[t + \hat{T}(t)\right]\left\{\sqrt{P_s}\, s\left[t + T(t)\right] + n(t)\right\} \tag{1}$$

where $\delta s(t) \overset{\Delta}{=} s(t + \Delta) - s(t - \Delta)$, and P_s is the received signal power. As will be shown, this product contains a low-pass spectral component which serves to keep the discriminator accurately tracking the target delay once the system has been "locked on." The low-pass filter which follows the cross-correlation network is designed on the basis of the expected dynamics of target motion, and its purpose is to remove as much noise and other interference as possible. The output of the loop filter is an estimate of the time derivative of delay (proportional to the target's radial velocity) and is used to control the clock rate of the FSR.

The delay estimate, $\hat{T}$, can be easily obtained by sensing the time instants when the transmitter and receiver FSRs go through a particular state, e.g., the "all ones" state, and by determining the time difference between these time instants.

III SYSTEM EQUATIONS

In order to analyze the response of the system, the signal cross-correlation term in (1) is expressed as

$$k\delta s(t + \hat{T})\, s(t + T)\sqrt{P_s} \overset{\Delta}{=} k\sqrt{P_s}\left[D_\Delta(\epsilon) + n_s(t, \epsilon)\right] \tag{2}$$

where $\epsilon \overset{\Delta}{=} T - \hat{T}$, is the delay error. The term which is not explicitly dependent on time, namely, $D_\Delta(\epsilon)$, the discriminator characteristic, has been separated from

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[5]Notice that $\delta s/2\Delta$ is reminiscent of the expression for the differentiated signal which is used in the cross-correlation operation in footnote reference 1:

$$\frac{ds}{dt} = \lim_{\Delta \to 0} \frac{s(t + \Delta) - s(t - \Delta)}{2\Delta}$$

the remainder, $n_s(t,\epsilon)$, the self-noise term. The discriminator characteristic can easily be shown to be[6]

$$\begin{aligned} D_\Delta(\epsilon) &= E\left[\delta s(t+\hat{T})s(t+T)\right] \\ &= R_s(\epsilon-\Delta) - R_s(\epsilon+\Delta) \end{aligned} \tag{3}$$

where $R_s(\quad)$ is the signal autocorrelation function.

In this paper the binary signal has amplitudes $s = \pm 1$. The autocorrelation function[7,8] and the discriminator characteristic for this maximal-length sequence are plotted in Fig. 2a, b. As can be seen, $D_\Delta(\epsilon)$ varies linearly with ϵ for $|\epsilon| < \Delta$, and is zero for $2\Delta \le |\epsilon| \le (M-2)\Delta$. Since $s(t)$ is periodic with period $M\Delta$, $D_\Delta(\epsilon)$ has the same periodicity.

The self-noise term in the output of the multiplier network can be expressed as

$$\begin{aligned} n_s(t,\epsilon) &= s(t+\Delta+\hat{T})s(t+T) - s(t-\Delta+\hat{T})s(t+T) - D_\Delta(\epsilon) \\ &= s(t+\Delta+\hat{T}) \oplus s(t+T) - s(t-\Delta+\hat{T}) \oplus s(t+T) - D_\Delta(\epsilon) \end{aligned} \tag{4}$$

where the symbol $\oplus$ represents modulo 2 addition. Multiplication of ± 1 terms is equivalent to modulo 2 addition.

In order to determine the effect of the self-noise on the system performance, it is necessary to determine the power spectrum of $n_s(t,\epsilon)$. This computation is

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[6] The symbol $E(\quad)$ represents the ensemble average operator. The sequence $s(t)$ is assumed to have an equidistributed random time origin.

[7] S. W. Golomb, "Sequences with randomness properties," Glenn Martin Co., Baltimore, Md., Final Report, Contract SC-54-36611, June, 1955.

[8] B. Elspas, "A radar system based on statistical estimation and resolution considerations," Stanford Electronics Lab., Stanford University, Stanford, Calif., Report #361-1, August, 1955, Appendix D.

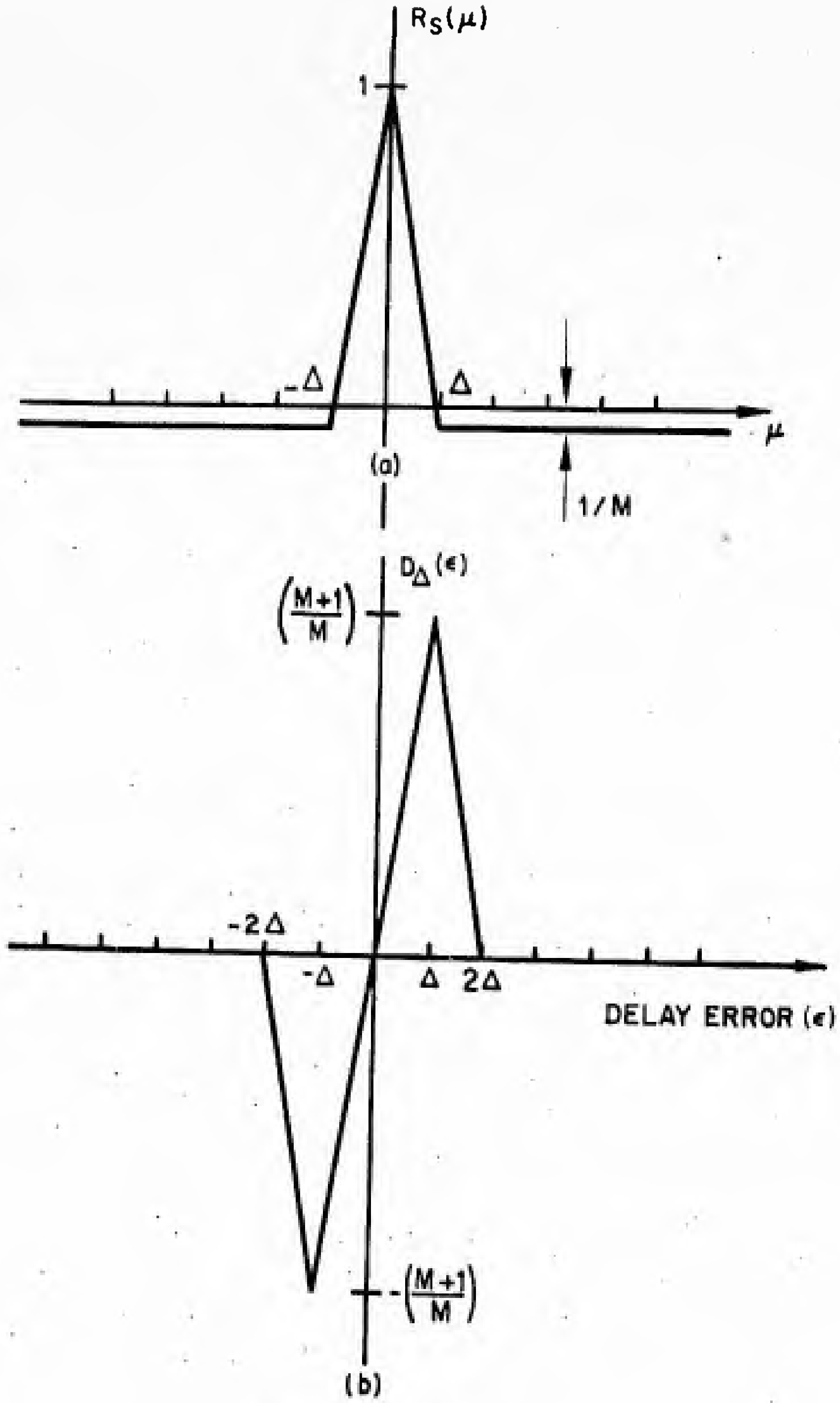

Fig. 2 Autocorrelation function, $R_S(\mu)$, and discriminator characteristic, $D_\Delta(\vec{\epsilon})$, for maximal-length binary shift-register sequences

performed for values of delay error $\epsilon = 0, \pm m\Delta$. It has been known for some time that maximal-length linear binary sequences possess the "cycle and add" property,[9,10] i.e.,[11] $s(t+i\Delta) \oplus s(t+j\Delta) = s(t+r\Delta)$. This relationship can be used to simplify the form of (4):

$$\begin{aligned} n_s(t,\epsilon = m\Delta) &= s(t+T+j\Delta) - s[(t+T+(j+1)\Delta] \quad , \text{ for } m = 0, M, \text{ etc.} \\ &= \pm s(t+T+n\Delta) \quad , \text{ for } m = \pm 1, M\pm 1, \text{ etc.} \\ &= s(t+T+r\Delta) - s(t+T+q\Delta) \quad , \text{ otherwise} \end{aligned} \tag{5}$$

where j, n, r, q are integers which are dependent on the delay error $\epsilon = m\Delta$; for no value of m is $r = q$. Hence, it is seen that for $\epsilon = \pm\Delta$ the self-noise power spectrum is the same as the signal power spectrum, $G_s(f)$. The signal power spectral density has components at frequencies which are integer multiples of $1/M\Delta$, i.e.,

$$G_s(f) = \frac{1}{M\Delta} G(f) \sum_{\nu=-\infty}^{\infty} \delta(f - \nu/M\Delta)$$

where $\delta(f - f_0)$ is the Dirac delta function defined by the operator equation, $\int H(f)\,\delta(f - f_0)\,df = H(f_0)$. The quantity $G(f)$, which can be considered as the "envelope" of the signal spectrum, is plotted in Fig. 3a.

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[9] S. W. Golomb, "Sequences with the cycle and add property," Jet Propulsion Laboratory, California Institute of Technology, Pasadena, Calif., Section Report #8-573, December, 1957.

[10] R. C. Titsworth, L. R. Welch, "Modulation by random and pseudo-random sequences," Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California, Progress Report #20-387, June, 1959, Sec. 3.

[11] In other words, if S is the state vector of the FSR at the time $t + i\Delta$, and T is the transformation matrix of the FSR, then $(I + T^{j-i})S = T^{r-i}S$ where I is the identity matrix. The relationship between i, j, r can be found using characteristic polynomial of T, i.e., $\phi(\lambda) = |T - \lambda I|$. In particular, one can use the Caley-Hamilton theorem, which shows that $\phi(T) = 0$, and the periodicity relation $T^M = I$, to obtain the relationship between these integers.

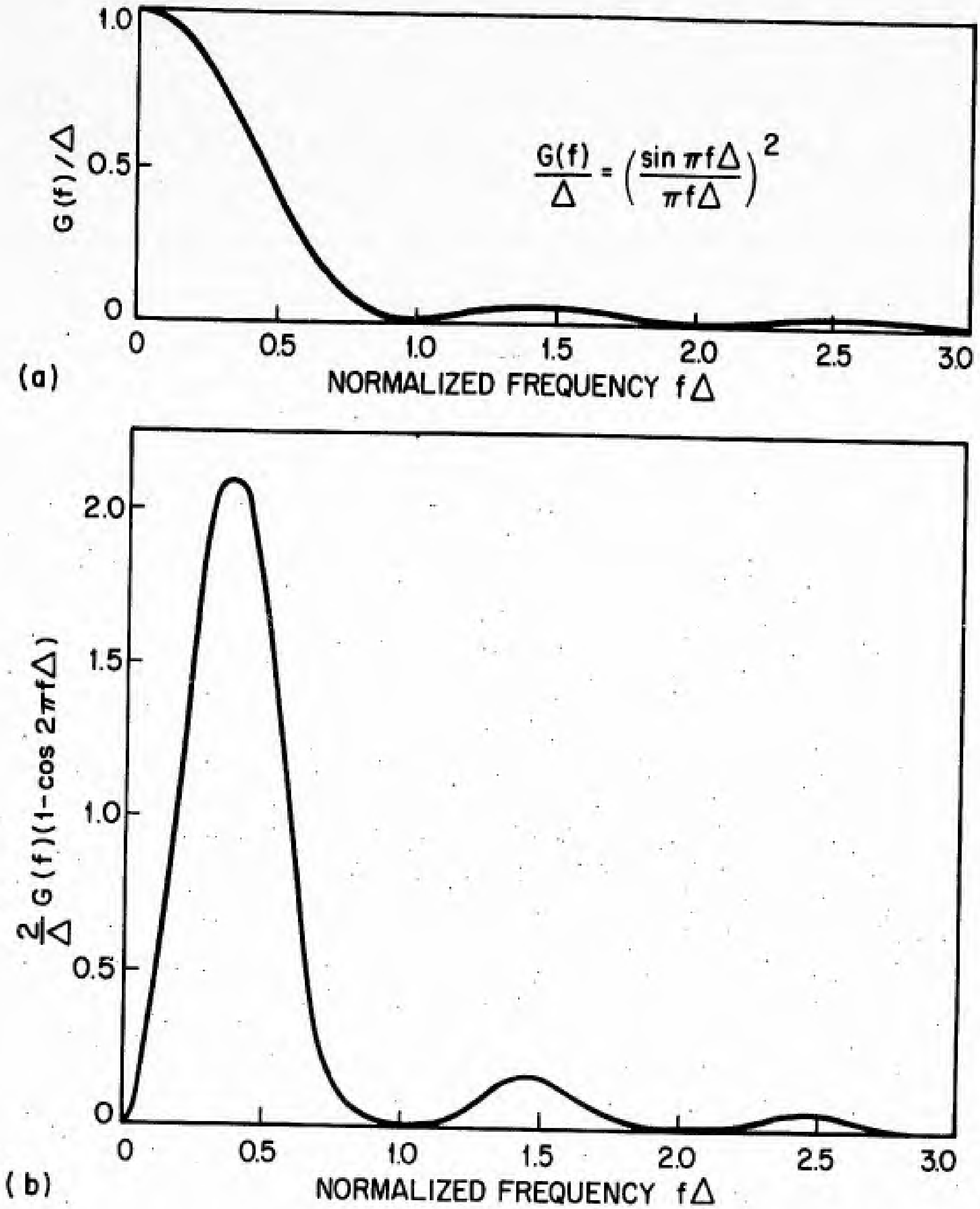

Fig. 3 "Envelopes" of power spectral densities, (a) binary sequence spectral density, (b) self-noise spectral density for $\epsilon = \Delta$

$$G(f) = \Delta\left(\frac{\sin \pi\Delta f}{\pi\Delta f}\right)^2$$

For other values of delay error, the self-noise is expressed as the difference between two versions of the same sequence having different time origins, and the self-noise spectrum is

$$\begin{aligned} G_{n_s}(f, \epsilon = m\Delta) &= 2G_s(f)\left[1 - \cos 2\pi f\Delta\right] &&, \text{ if } m = 0\ , M\ , \text{etc.} \\ &= 2G_s(f)\left[1 - \cos 2\pi(r - q)f\Delta\right] &&, \text{ if } 1 < |m| < M - 1\ , \text{etc.} \end{aligned} \tag{6}$$

The self-noise spectrum for $\epsilon = 0$ is depicted in Fig. 3b. Notice that it is a general characteristic of the spectra in (6) that they have a null at the origin, and as a result the self-noise is more easily removed by the loop filter.

The noise input to the discriminator is assumed to be white Gaussian noise. Because of this spectral density, it produces a noise component in the output of the cross-correlation network which is also white

$$k\delta s(t + \hat{T})n(t) \triangleq kn_n(t) \tag{7}$$

The spectral density of $n_n(t)$ is $G_{n_n}(f) = P_d N_o$, where P_d is the average power of $\delta s(t + \hat{T})$ and N_o is the input noise spectral density. For values of $\hat{T}$ which are fixed or vary little in a time interval Δ, the average power in $\delta s(t + \hat{T}) \triangleq s(t + \hat{T} + \Delta) - s(t + \hat{T} - \Delta)$ is $P_d = 2$. Hence, $G_{n_n}(f) = 2N_o$ watt-sec.

Thus, the output of the cross-correlator network can be written, using (1), (2), and (7) as

$$k\sqrt{P_s}\left[D_\Delta(\epsilon) + n_s(t, \epsilon) + n_n(t)/\sqrt{P_s}\right] \tag{8}$$

By referring to Figs. 1 and (8), it can be seen that the system equation (in operator notation) is

$$p\hat{T} = kg_c g_f \Delta \sqrt{P_s}\; F(p/p_0) \left[D_\Delta(\epsilon) + n_s(t,\epsilon) + n_n(t)/\sqrt{P_s} \right] \tag{9}$$

where g_f is the loop-filter gain-constant, p_0 is the loop-filter frequency-constant, and g_c is the gain of the voltage-controlled clock. Define g_0 as the dc loop gain, i.e., a steady delay error of ϵ sec. $(|\epsilon| < \Delta)$ produces a $g_0\epsilon/\Delta$ cps change in clock frequency or a $g_0\epsilon$ sec. delay change/sec. The dc loop gain is given by $g_0 = kg_f g_c \sqrt{P_s}\,(M + 1)/M$. The system equation, (9), can then be written as

$$p\hat{T} = g_0 \Delta\, F(p/p_0) \left(\frac{M}{M + 1}\right) \left[D_\Delta(\epsilon) + n_s(t,\epsilon) + n_n(t)/\sqrt{P_s} \right] \tag{10}$$

VI' NOISE PERFORMANCE

In this section the effects of receiver noise and self-noise on system accuracy are determined. The discriminator is assumed to be in the locked-on state, i.e., $|\epsilon| < \Delta$, so that the system is operating in the region where $D_\Delta(\epsilon) = [(M + 1)/M]\,(\epsilon/\Delta)$.

Define the normalized loop gain $g = g_0/p_0$. The linearized system equation can then be obtained from (10) as

$$\hat{T}/\Delta = g\, \frac{F(p/p_0)}{(p/p_0)} \left[\frac{\epsilon}{\Delta} + \frac{n_s(t,\epsilon) + n_n(t)/\sqrt{P_s}}{(M + 1)/M} \right]$$

Since $\epsilon = T - \hat{T}$, (10) can be written as

$$\hat{T}/\Delta = H(p/p_0)\left[T/\Delta + \frac{n_s(t,\epsilon) + n_n(t)/\sqrt{P_s}}{(M+1)/M}\right] \tag{11}$$

where $H(p/p_0)$ is the linearized closed-loop transfer function

$$H(p/p_0) \triangleq \frac{g\,F(p/p_0)}{p/p_0 + g\,F(p/p_0)}$$

The linear feedback network depicted in Fig. 4a can be shown to have the same system equation as (11), and therefore it provides performance equivalent to that of Fig. 1 when the equivalent input $\frac{T(t)}{\Delta} + \frac{n_s(t,\epsilon) + n_n(t)/\sqrt{P_s}}{(M+1)/M}$ is applied, and $|\epsilon| < \Delta$.

Consider that the closed-loop transfer function $H(p/p_0)$ has the form

$$H(p/p_0) = \frac{1 + \sqrt{2}\,p/p_0}{1 + \sqrt{2}\,p/p_0 + (p/p_0)^2}$$

This transfer function has been shown optimum for ramp inputs of delay in the presence of white noise in that it minimizes the total squared transient error plus the mean squared error caused by interfering noise.[12] The relationship between p_0 and the transient response is discussed further in the next section. The loop-filter shown in Fig. 4b .

[12] R. Jaffe, E. Rechtin, "Design and performance of phase-locked circuits capable of near-optimum performance over a wide range of input signal levels," IRE Trans., vol. IT-1, pp. 66-72, March, 1955.

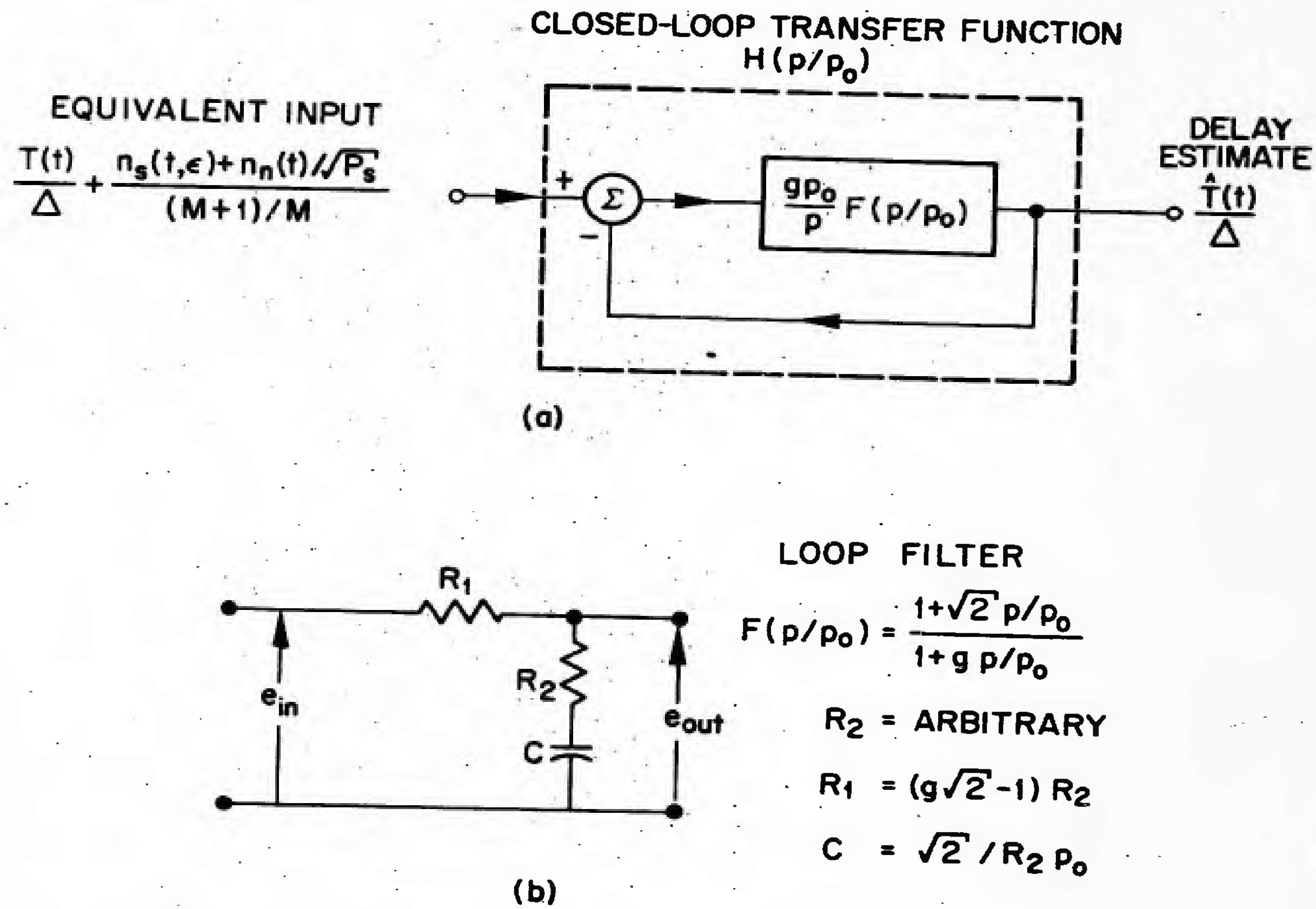

Fig. 4 Linearized equivalent circuit for the delay-lock discriminator, valid for $|\epsilon| < \Delta$, (a) block diagram, (b) loop-filter referred to in the analysis

$$F(p/p_o) = \frac{1 + \sqrt{2}\, p/p_o}{1 + g p/p_o} \tag{12}$$

can be used to approximate this closed-loop response as closely as desired by making the normalized gain constant g large.

The mean square delay error caused by the receiver noise can be evaluated from (7) and (11) to be

$$\sigma_{n_n}^2 = \frac{\Delta^2}{P_s}\left(\frac{M}{M+1}\right)^2 \int_{-\infty}^{\infty} G_{n_n}(f)\,|H(j\omega/p_o)|^2\, df = 2.12\Delta^2\left(\frac{M}{M+1}\right)^2 \frac{N_o p_o}{P_s} \tag{13}$$

For example, if the receiver noise temperature is 900°K, $p_o = 1$ rad/sec., $M \gg 1$, $\Delta = 10^{-6}$ sec. and $P_s = 10^{-15}$ watt, then $N_o = 6.2 \times 10^{-21}$ watt-sec., and the rms delay error caused by noise is $\sigma_{n_n} = 5.3 \times 10^{-9}$ sec.

The self-noise spectrum has been shown to depend upon the delay error ϵ. Hence, the mean square value of delay error caused by the self-noise also depends on ϵ. An upper bound on the mean square delay error can be found by assuming the most pessimistic spectral density function, i.e., the self-noise spectrum obtained for $\epsilon = \pm\Delta$. The mean square self-noise delay error is then

$$\sigma_{n_s}^2 = \Delta^2 \int_{-\infty}^{\infty} G_{n_s}(f)\,|H(j\omega/p_o)|^2\, df = \frac{\Delta^2}{M} \sum_{\nu=-\infty}^{\infty} \frac{1 + 2(2\pi\nu/M\Delta p_o)^2}{1 + (2\pi\nu/M\Delta p_o)^4} \left(\frac{\sin \nu\pi/M}{\nu\pi/M}\right)^2$$

Under conditions where the number of digits in a period M is large and the loop filter constant p_o is small compared to the signal bandwidth ($\approx 1/2\Delta$), this expression simplifies to

$$\sigma_{n_s}^2 \cong \Delta^2\left(\frac{p_o\Delta}{2}\right) \tag{14}$$

For the parameters just considered, $p_o = 1$ rad/sec., $\Delta = 10^{-6}$ sec., (14) gives an rms self-noise delay error of $\sigma_{n_s} = 7.07 \times 10^{-10}$ sec.

When the delay error fluctuations become too large, the discriminator encounters a threshold effect. This effect is caused by the fact that large delay errors force the discriminator to operate, at least temporarily, in a region of negative loop gain, since $D(\epsilon)$ has a negative slope for $\Delta < |\epsilon| < 2\Delta$.

Experimental measurements made thus far indicate that the system threshold occurs when the total rms delay error is about $\sigma_n \doteq 0.30\Delta$. For delay error having $\sigma_n \leq 0.30\Delta$, experiments showed a negligible probability of losing the locked-on state in the absence of transient errors. For delay error fluctuations which have a Gaussian amplitude statistic and $\sigma_n = 0.30\Delta$, the probability that $|\epsilon| > \Delta$ in the linearized model is only 0.00087. This statement assumes that transient errors caused by fluctuations in T are small compared to Δ. Assuming that the self-noise error is negligible, the system threshold can be computed from (13) and is given approximately by

$$\frac{P_s}{p_o N_o} = 7.06\left(\frac{M}{M+1}\right)^2 \tag{15}$$

V TRANSIENT PERFORMANCE

In this section two problems concerning the transient performance of the discriminator in the absence of noise are investigated:

1. How rapidly can a given region be searched and the target acquired for a given closed-loop noise bandwidth, $B_n = 1.06\ p_o$ cps?
2. Once the target has been acquired, what is the maximum change in velocity that can be tolerated without losing the locked-on condition?

In order to solve these problems most readily, we rewrite (10), neglecting the noise effects, and making use of the following time normalizations $x \triangleq \epsilon/\Delta$, $y \triangleq T/\Delta$, $y - x = T/\Delta$, $\tau \triangleq p_0 t$, $s \triangleq p/p_0 = d/d\tau$, $D(x) \triangleq D_\Delta(\epsilon)$

$$s(y - x) = gF(s)D(x) \tag{16}$$

The integral plus proportional control loop-filter of (12) now becomes

$$gF(s) = \frac{1 + \sqrt{2}\,s}{1/g + s}$$

Thus we obtain the operator equation

$$(1/g + s)s(y - x) = (1 + \sqrt{2}\,s)D(x)$$

This system equation can be rewritten in time-derivative notation

$$\dot{y}/g + \ddot{y} = \dot{x}/g + \ddot{x} + D(x) + \sqrt{2}\,D'(x)\dot{x} \tag{17}$$

where $\dot{x} = dx/d\tau$, $D'(x) = dD/dx$, etc.

The phase-plane method of solving this second-order differential equation is to compute the trajectories in x , $\dot{x}$ space which describe the solution of this equation for the desired sets of initial conditions. Define a new variable $\gamma \triangleq \ddot{x}/\dot{x} = d\dot{x}/dx$, which from (17) is given by

$$\frac{d\dot{x}}{dx} = \gamma[x , \dot{x} , \dot{y} , \ddot{y}] = - \frac{D(x) + [\sqrt{2}\,D'(x) + 1/g]\dot{x} - \dot{y}/g - \ddot{y}}{\dot{x}} \tag{18}$$

Computer solutions to these trajectories can be obtained by approximating the differential equation (18) with the difference equation

$$\dot{x}_{n+1} - \dot{x}_n \triangleq \dot{x}\left(x_o + \sum_{j=0}^{n} \delta j\right) - \dot{x}\left(x_o + \sum_{j=0}^{n-1} \delta_j\right) \cong \gamma(\dot{x}_n, x_n)\delta_n \qquad (19)$$

where x_o, $\dot{x}_o$ are the initial values of x, $\dot{x}$, and δ_j is the j-th increment in x. Satisfactory solutions can be obtained by letting the computer function as an adaptive device so that the size of the interval is made variable. In this particular sequential computation[13] the interval size is taken as[14]

$$|\delta_j| = \frac{\delta}{1 + |\gamma(\dot{x}_j, x_j)|}$$

The value δ to be used is 0.02.

[13] R. Bellman, Adpative Control Processes: A Guided Tour, Princeton University Press, Princeton, New Jersey, 1961, Chapter 4.

[14] Notice that the distance moved in the x, $\dot{x}$ plane in one increment is

$$\Delta r_n = \sqrt{(x_{n+1} - x_n)^2 + (\dot{x}_{n+1} - \dot{x}_n)^2} = \delta\left(\frac{1 + \gamma_n^2}{1 + 2|\gamma_n| + \gamma_n^2}\right)^{1/2}$$

Hence, the distance moved is bounded by $\delta/\sqrt{2} \le r_n \le \delta$. Thus, the maximum distance moved is δ, yet needlessly small increments are not used for small or moderate $|\gamma|$.

Consider the search and acquisition problem where the normalized loop gain $g = \infty$. Assume that the target search velocity is a constant, $\dot{y}(t) = \dot{y}$, $\ddot{y} = 0$. The variable γ now becomes

$$\gamma(x, \dot{x}) = -\frac{D(x) + \sqrt{2}\, D'(x)\dot{x}}{\dot{x}}$$

The acquisition trajectories for these conditions are plotted[15] in Fig. 5. If the delay error is decreasing from the left the system does not respond until $x = -2$. As can be seen, if the search velocity is $|\dot{y}| \leq 2.2$, the system locks on and the state variable converges to the origin. As an example, consider $\Delta = 10^{-6}$ sec, $p_0 = 10$ rps. Then we have $\dot{y} = (dT/dt)/p_0\Delta = 10^5\, dT/dt$. For electromagnetic propagation $c = 3 \times 10^5$ km/sec. and the maximum search velocity (two-way propagation time) permitted is $v = (1/2)\, 2.2cp_0\Delta = 3.3$ km/sec. From (15), it can be seen that the threshold value of signal power is $P_s = 6.36\, N_0 v/c\Delta = 70.6\, N_0$ watt in this example $(\dot{M} \gg 1)$.

Notice, however, that even if the system fails to lock-on in this particular interval of x, the velocity error is less at the end of the transient than at the beginning. Since $D(x)$ is periodic every M, the system always locks on eventually because with $g = \infty$ there is no decay in $\dot{x}$ outside of the intervals $|x| < 2$. It may, however, pass through stable regions of x many times before locking on. In practice, this behavior might rely on unrealistic storage times in the filter, as is implicit in the assumption that $g = \infty$. Furthermore, the time required for lock-on might be intolerable for search velocities $|\dot{y}| > 2.2$.

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[15] The trajectory computations were performed by Lt. C. S. Mulloy on a CDC 1604 computer and are presented as part of his report entitled "Digital analysis of the delay-lock discriminator," U. S. Naval Postgraduate School, Monterey, California, 1962.

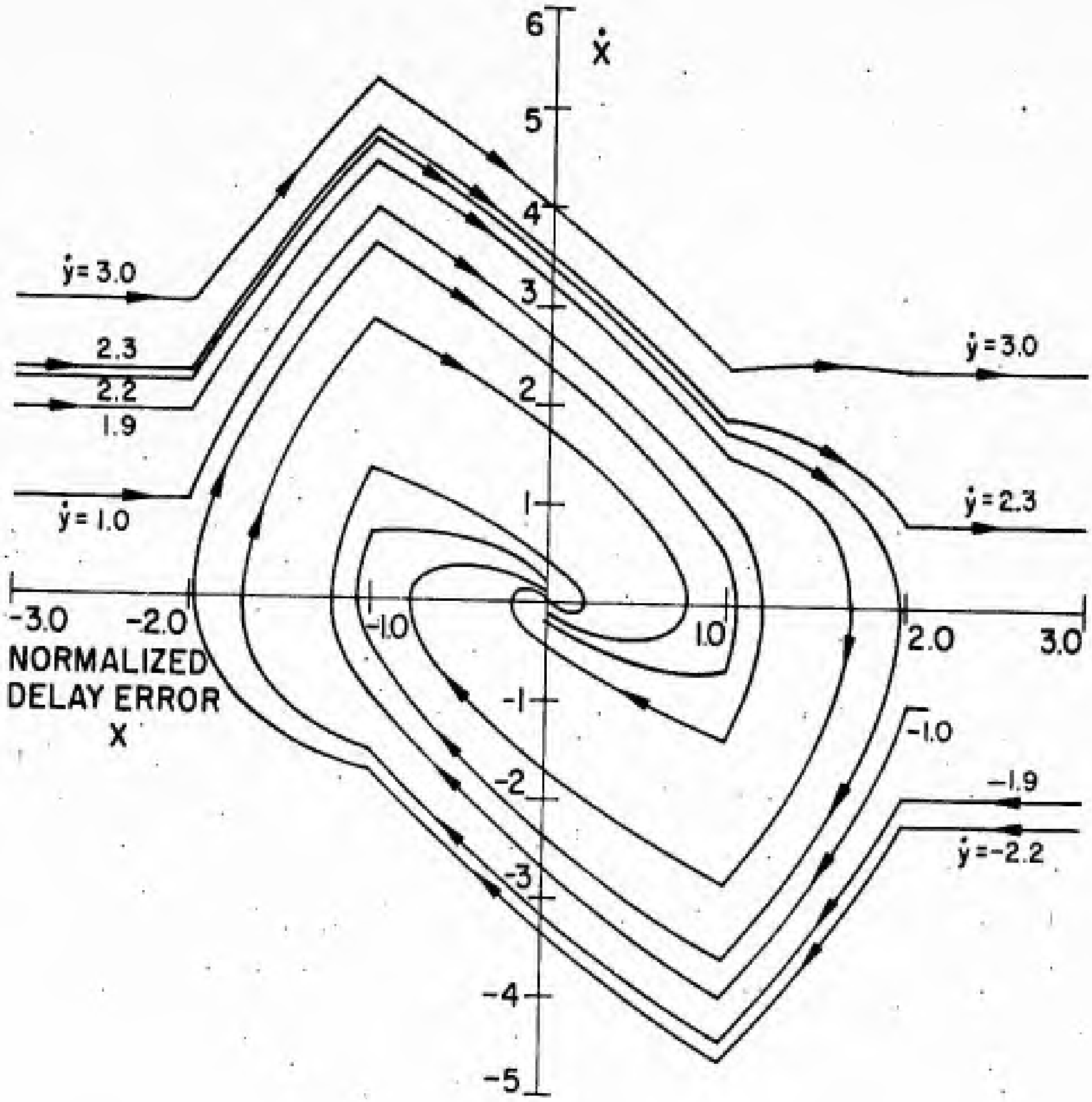

Fig. 5 Acquisition trajectories for loop gain $g = \infty$, plotted for various values of $\dot{y}$, the normalized search velocity

The effect of target velocity transients can also be ascertained from Fig. 5. If the system is initially locked on, $x = 0$, $\dot{x} = 0$ and the target suddenly changes velocity to $\dot{y}$, the system response is described by that portion of the trajectories which begins at $x = 0$, $\dot{x} = \dot{y}$. As can be seen, normalized velocity transients of $\dot{y} = 3.38$ can be tolerated without losing the locked-on state.

Acquisition trajectories have also been obtained for finite loop gain systems with $g = 10$. The equation for γ in this example is

$$\gamma(x, \dot{x}, \dot{y}) = - \frac{D(x) + (\sqrt{2}\, D'(x) + 0.1)\dot{x} - 0.1\dot{y}}{\dot{x}}$$

Because of the finite loop gain, the maximum steady-state clock-frequency change is $g = 10\,p_0$. Thus, for this reason alone, the system would never lock on for $|\dot{y}| > 10$.

The trajectories for these conditions are shown in Fig. 6. As can be seen, the fact that the loop gain is reduced from ∞ to 10 has very little effect on the maximum tolerable search velocity; $|\dot{y}| \leq 2.2$ is still permitted. Notice however, that the curves which lock on converge to $x = 0.1\dot{y}$ or $\epsilon = x\Delta = 0.1\Delta\dot{y}$ rather than to $x = 0$. For $\Delta = 10^{-6}$ sec. and $\dot{y} = 1$, the steady-state lock-on point is $\epsilon = 0.1$ microseconds. In practice, by measuring $\dot{y}$ one can correct for this steady-state bias error.

Although it is not shown on the trajectory curves, the value of $\dot{x}$ for the curves which do not lock on decays with a time constant $g/p_0 = 10/p_0$ sec. Thus, if the period of the sequence is large so that $M \gg 1$, the value of $\dot{x}$ will have decayed almost to $\dot{y}$ when $x = M - 2$.

As well as knowing whether or not the system will lock on for a given search velocity, it is also important to know how long the transient lasts. The time response of the system can be obtained from computer solutions to the difference equation

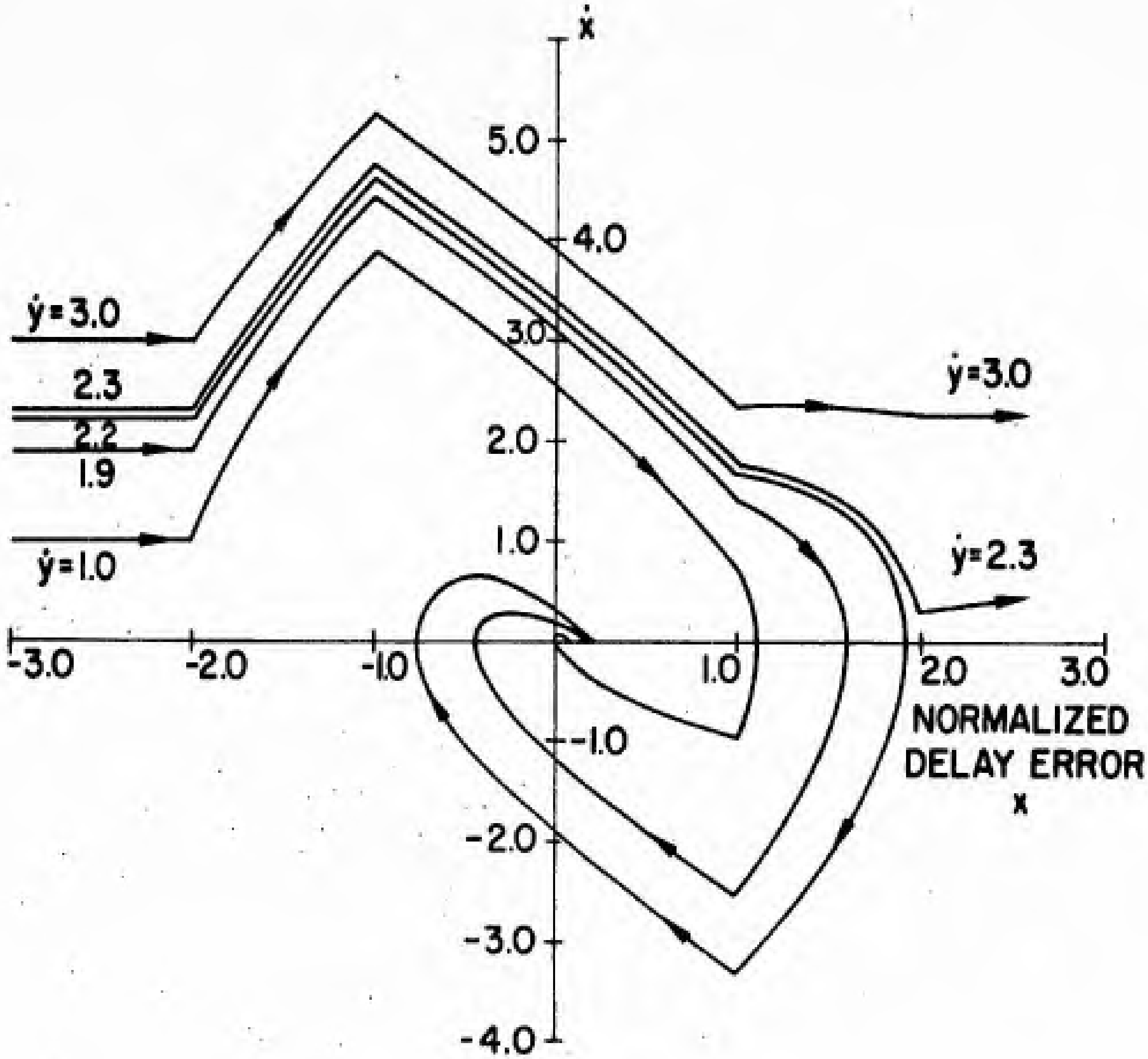

Fig. 6 Acquisition trajectories for loop gain $g = 10$, plotted for various values of $\dot{y}$, the normalized search velocity

$$t_{n+1} - t_n \triangleq t\left(x_o + \sum_{i=0}^{n} \delta_i\right) - t\left(x_o + \sum_{i=0}^{n-1} \delta_i\right) \cong \frac{\delta}{(1 + |\gamma_n|)\dot{x}(x_n)}$$

The results of this computation are shown in Figs. 7a, b as plots of x and $\dot{x}$ for $g = \infty$; the maximum tolerable search velocity is $\dot{y} = 2.2$. The time required for the transient to subside within $|x| < 0.1$ is $\tau = 5.6$. For $p_o = 10$ rad/sec. this lock-on time is about 0.56 sec.

VI EFFECT OF AMPLITUDE-LIMITING THE RECEIVED DATA

Considerable simplification in the circuitry is possible if the received data are amplitude-limited before entering the cross-correlation network. The input to the cross-correlation network then has the binary form $u(t) = A \operatorname{sgn}\left[\sqrt{P_s}\, s(t) + n(t)\right]$ where A is the limiter output amplitude and $\operatorname{sgn} x$ is the sign function

$$\begin{aligned} \operatorname{sgn} x &= 1 \ , \ \text{if } x \geq 0 \\ &= -1 \ , \ \text{if } x < 0 \end{aligned}$$

Thus, the waveforms fed into the discriminator have $\pm A$ amplitudes, and, as a result, the multiplication circuits of Fig. 1 can be replaced by modulo 2 adders, as shown in Fig. 8. It is apparent that this change is inconsequential for a noise-free input. The purpose of this section is to determine how much this limiting action influences the noise performance when the input signal-to-noise ratio (SNR) is small.

The first step in the solution of this problem is to compute the cross-correlation between the output of the limiter $u(t + \mu)$ and the signal component $s(t)$. It is this cross-correlation which determines the useful signal component in the correlator network output through its relationship to the discriminator characteristic. Consider that

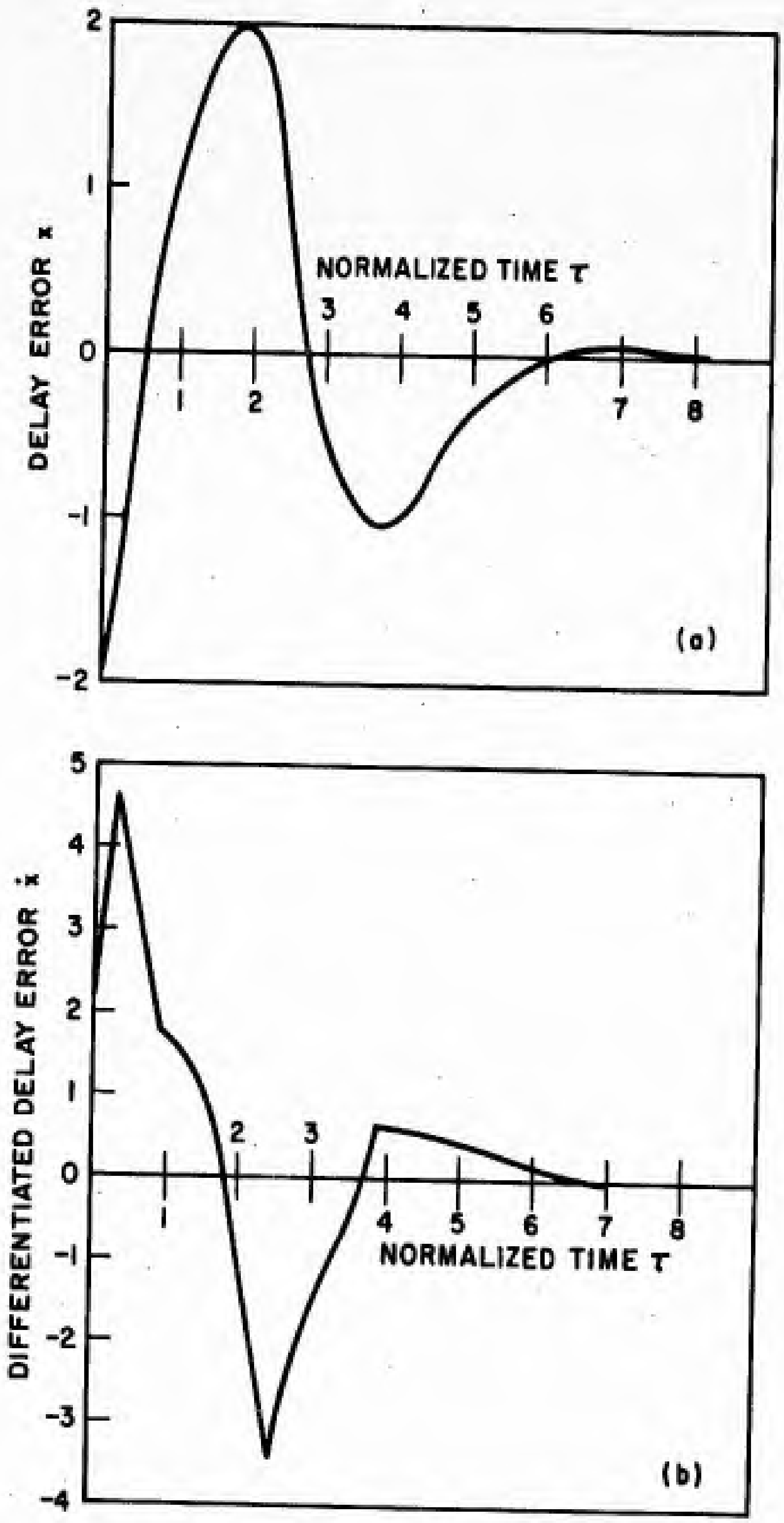

Fig. 7 Acquisition transient for $g = \infty$ and maximum search velocity, $\dot{y} = 2.2$, (a) normalized delay error response, (b) differentiated delay error response

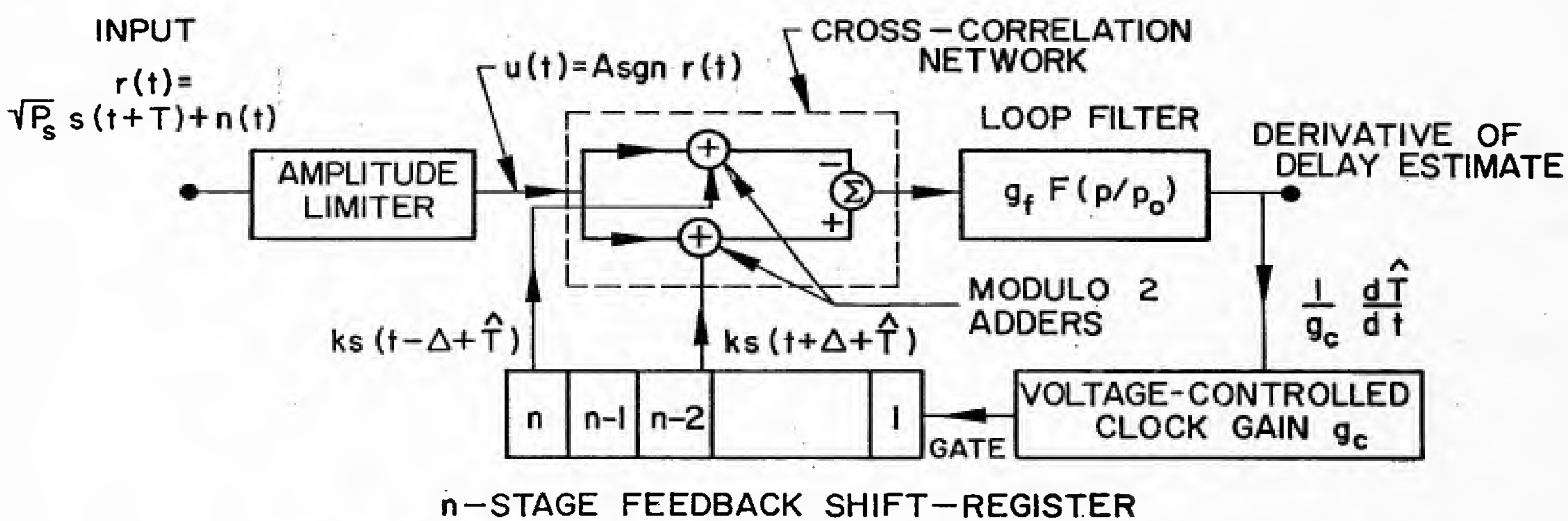

Fig. 8 Delay-lock discriminator operating on amplitude-limited inputs

bandlimited white Gaussian noise enters the limiter. The noise is low-pass having a maximum frequency B and average power P_n. It is shown in the appendix that the cross-correlation is related to the signal autocorrelation function by the equation

$$E\left[u(t+\mu)s(t)\right] \triangleq AR_{us}(\mu) = AR_s(\mu)\ \mathrm{erf}\sqrt{P_s/2P_n} \tag{20}$$

where erf x is the error function. For small input SNR, (20) becomes

$$AR_{us}(\mu) = \sqrt{\frac{2}{\pi}\frac{P_s}{P_n}}\ AR_s(\mu)$$

It is interesting to compare this cross-correlation function with that which is obtained without the limiter when the average total input power is held constant at the same level A^2. This type of operation is obtained if the discriminator is preceded by an "ideal" AGC amplifier which has constant average output power A^2. The cross-correlation for this situation is

$$AR_s(\mu)\sqrt{\frac{P_s}{P_s + P_n}}$$

By comparing this expression with (20), it is apparent that at small values of input SNR, the cross-correlation with amplitude-limited inputs is smaller by a factor of $\sqrt{2/\pi}$. Since both expressions are proportional to the square root of the input SNR for small SNR, the loop gain of the discriminator, g_o, decreases in the same proportion. More generally, the loop gain with limiting, g_{oL}, varies with SNR as

$$g_{oL} = g_o\ \mathrm{erf}\sqrt{P_s/2P_n}$$

where g_o is the loop gain for noise-free inputs. This expression is plotted in Fig. 9.

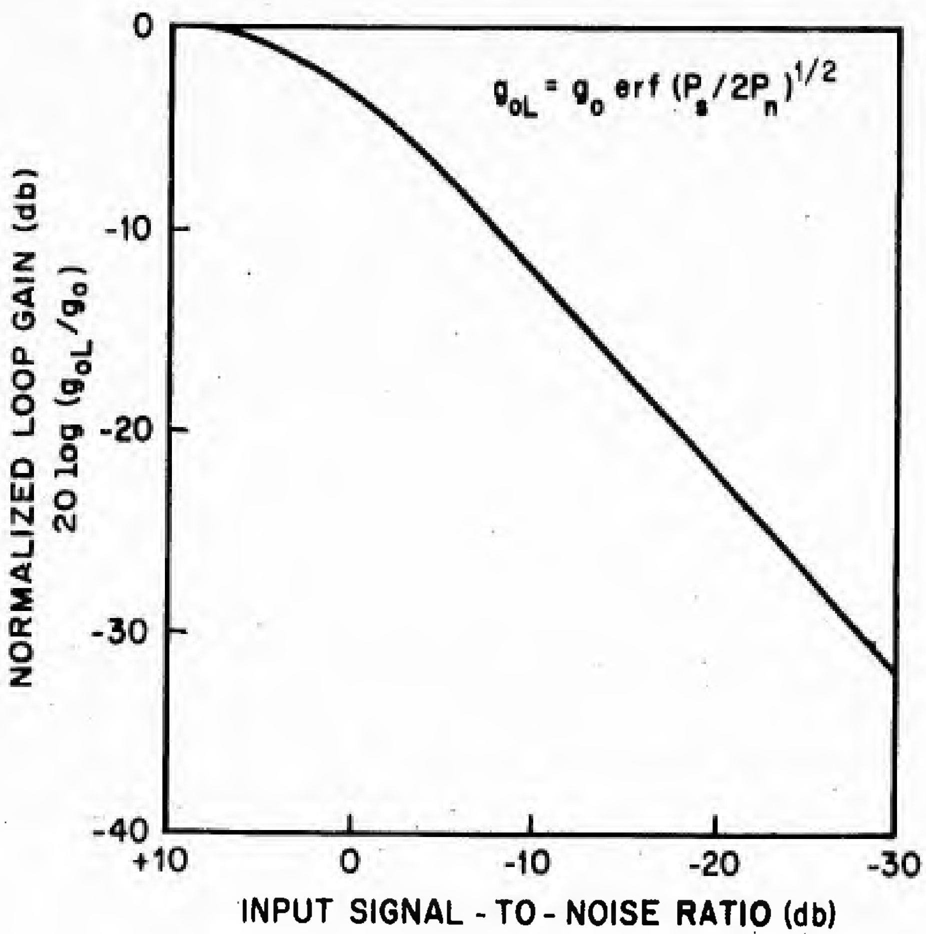

Fig. 9 Normalized loop gain with limiting vs. input SNR

It is next necessary to determine the spectral density of the noise output of the cross-correlator network. It is known that at low SNR the autocorrelation function of the noise in the limiter output is[16]

$$R_{no}(\mu) = \frac{2}{\pi} A^2 \sin^{-1} \frac{R_n(\mu)}{P_n}$$

Hence, the noise spectrum in the limiter output is[17]

$$G_{no}(f) = \frac{2}{\pi} \frac{A^2}{P_n} \left[G_n(f) + 0.167\ {}_3G_n(f)/P_n^2 + 0.075\ {}_5G_n(f)/P_n^4 + \ldots \right]$$

Assume that the noise bandwidth is large compared to the signal bandwidth. Then the noise effects in the cross-correlator output are dependent mainly upon $G_{no}(0)$, and this spectral density has the value

$$G_{no}(0) = 2 \int_0^\infty \frac{2}{\pi} A^2 \sin^{-1} \left[R_n(\mu)/P_n \right] d\mu = \frac{4}{\pi} A^2 \int_0^\infty \sin^{-1} \left[\frac{\sin 2\pi B\mu}{2\pi B\mu} \right] d\mu$$

$$= \frac{2}{\pi} \frac{A^2}{\pi B} \int_0^\infty \left[\frac{\sin x}{x} + \frac{1}{6} \left(\frac{\sin x}{x} \right)^3 + \frac{3}{40} \left(\frac{\sin x}{x} \right)^5 + \ldots \right] dx = \frac{A^2}{2B} \left(\frac{2.2}{\pi} \right) \tag{21}$$

The degradation in system performance at low SNR caused by the limiter action can now be determined by taking the ratio of the square of the signal component amplitude

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[16] R. Price, "A useful theorem for nonlinear devices having Gaussian inputs," IRE Trans., vol. IT-4, pp. 69–72, June, 1958.

[17] The notation ${}_3G_n(f)$ represents the convolution in the frequency domain $G_n(f)*G_n(f)*G_n(f)$.

at the cross-correlator output to the noise spectral density $G_{no}(0)$. This ratio is then compared with the corresponding ratio obtained with an ideal AGC amplifier having average output power A^2 . For operation with the limiter, this ratio can be obtained from (20), (21) as

$$\frac{R_{us}^2(\mu)/R_s^2(\mu)}{G_{no}(0)} = \frac{P_s}{P_n/2B}\left(\frac{1}{1.1}\right) \tag{22}$$

Operation with the ideal AGC yields a ratio which is larger by only a factor of 1.1 or 0.4 db. Hence the use of an amplitude limiter is attractive, since it allows the use of binary logic for the cross-correlation circuitry at only a small expense in the theoretical noise performance.

VII DISCUSSION

The transient and noise analyses show that the delay-lock tracking system possesses several features which are desirable for long-range tracking operations. For example, the system threshold occurs at a SNR which permits operation at very low power levels even for rather large target velocities. At the same time, the signal can be constructed to have periodicity long enough to resolve any range ambiguities which might otherwise occur for large target distances.

Solution of the phase-plane trajectories has demonstrated that the system is capable of reasonably large search velocities, relative to the loop-filter time-constant, which result in target acquisition. It should be pointed out here that by making the loop-filter adaptive, certain advantages in noise performance can result. One can sense, by means of an external correlator, whether or not the system is locked on. When the loop is unlocked and the discriminator is in its search mode, the loop-filter can be designed to permit target acquisition with the desired search velocity. Once the target

has been acquired, the closed-loop bandwidth can be decreased, and the noise performance of the loop can thereby be improved. More extensive discussion of this potential will be presented in a later paper.

Finally, it has been shown that amplitude-limiting the received data permits the use of binary logic in place of analog multipliers at only a small expense in the theoretical noise performance. In practice, deficiencies in performance of the analog circuitry would often more than make up for this theoretical difference. With either analog or binary inputs to the discriminator, if the average input power is fixed, the loop gain decreases in proportion to the square root of the SNR, and the closed-loop bandwidth and maximum search velocity are diminished accordingly. Hence, the loop gain should be made large enough to satisfy the search velocity requirements at the lowest SNR likely to be encountered.

ACKNOWLEDGMENT

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APPENDIX

The purpose of this appendix is to compute the cross-correlation between the limiter output $u(t+\mu)$ and the reference signal component $s(t)$. This cross-correlation can be written

$$E\left[z(t) \triangleq \frac{u(t+\mu)s(t)}{A}\right] = R_{us}(\mu) = \Pr\left[z(t) = 1\right] - \Pr\left[z(t) = -1\right]$$

where $u(t) = A \operatorname{sgn}\left[\sqrt{P_s}\, s(t) + n(t)\right]$. In order to simplify the notation, define $s(t) = s_o$, $s(t+\mu) = s_\mu$, $n(t+\mu) = n_\mu$. Define the signal probabilities $\Pr(s = 1) = p = (M+1)/2M$, and $\Pr(s = -1) = q = 1 - p$. The probability that $z = 1$ can then be written

$$\Pr(z = 1) = p\Pr(n_\mu > -\sqrt{P_s}\, s_\mu | s_o = 1) + q\Pr(n_\mu < -\sqrt{P_s}\, s_\mu | s_o = -1) \tag{23}$$

$$= \Pr(n_\mu > -\sqrt{P_s})\left[p\Pr(s_\mu = 1 | s_o = 1) + q\Pr(s_\mu = -1 | s_o = -1)\right]$$

$$+ \Pr(n_\mu > \sqrt{P_s})\left[p\Pr(s_\mu = -1 | s_o = 1) + q\Pr(s_\mu = 1 | s_o = -1)\right]$$

where use has been made of the fact that the probability density of n is symmetric about the origin. The probability that $z = 1$ can be obtained in a similar manner

$$\Pr(z = -1) = \Pr(n_\mu < -\sqrt{P_s})\left[p\Pr(s_\mu = 1 | s_o = 1) + q\Pr(s_\mu = -1 | s_o = -1)\right]$$

$$+ \Pr(n_\mu < \sqrt{P_s})\left[p\Pr(s_\mu = -1 | s_o = 1) + q\Pr(s_\mu = 1 | s_o = -1)\right] \tag{24}$$

The autocorrelation function of the signal is expressed by

$$R_s(\mu) = E\left[s(t)s(t+\mu)\right] = p\left[\Pr(s_\mu = 1|s_o = 1) - \Pr(s_\mu = -1|s_o = 1)\right]$$

$$+ q\left[\Pr(s_\mu = -1|s_o = -1) - \Pr(s_\mu = 1|s_o = -1)\right]$$

Hence, by using (23), (24), the cross-correlation function $R_{us}(\mu)$ can be written

$$R_{us}(\mu) = R_s(\mu)\Pr(|n_\mu| < \sqrt{P_s})$$

Under the assumption that the noise has stationary Gaussian amplitude statistics and has a mean square value P_n, we obtain

$$R_{us}(\mu) = R_s(\mu)\,\mathrm{erf}\sqrt{P_s/2P_n}$$

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